THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 51, No. 6, July-August, 1972
Printed in U.S.A.

Dimensionality of Crosstalk Functions

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(Manuscript received February 7, 1972)

We study the dimensionality for the class of near-end crosstalk functions in a cable. The dimensionality is closely related to the distribution of eigenvalues for a particular integral operator that we call the energy operator. We find bounds for these eigenvalues in terms of the eigenvalues associated with the prolate spheroidal waveforms studied by Landau, Pollak, and Slepian. The important technical observation, permitting us to use their results, is that though the crosstalk functions are not bandlimited, the degree to which they are band-concentrated can be uniformly specified.

I. INTRODUCTION

The class of functions bandlimited to the interval $(-W, W)$ and considered over the interval $(-T, T)$ has long been held to have essentially $[2WT]$ degrees of freedom. This goes back at least as far as the discovery of the sampling (or cardinal) series, since exactly this number of terms in the series is available with knowledge of the function over the interval $(-T, T)$.[1] The notion was made precise and validated by Landau and Pollak.[2] The fundamental quantity in their approach was the energy (or L^2-norm) of the bandlimited function over $(-T, T)$. The energy is computed as a quadratic form of the function and to this there corresponds a positive definite, compact operator. We shall call this the energy operator. The distribution of the eigenvalues for the energy operator, i.e., the energy eigenvalues, determine the approximate number of degrees of freedom or dimensionality of this class of functions. The idea is that energy eigenfunctions with small enough eigenvalues (or energy) can contribute only minimally to the energy in the interval; hence, they can be disregarded. They find that $[2WT]$ energy eigenfunctions span this space of functions within an error bound which they compute.

To be more definite, let D_T denote the operator which acts on square integrable functions as follows:

$$D_T f(t) = \begin{cases} f(t) & t \varepsilon (-T, T) \\ 0 & t \not\varepsilon (-T, T) \end{cases}$$

and let B_W denote the operator which similarly chops off the Fourier transform of the function outside $(-W, W)$; thus, if $F(\omega)$ is the Fourier transform of $f(t)$, then

$$B_W f(t) = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} e^{i\omega t} F(\omega) \, d\omega.$$

Whenever f is bandlimited to $(-W, W)$, then $B_W f = f$. The energy of this function in the interval $(-T, T)$ is

$$\| D_T f \|^2 = \| D_T B_W f \|^2 = (D_T B_W f, D_T B_W f) = (f, D_T B_W f),$$

where we have made use of the fact that D_T is a projection operator and $B_W f = f$. Note:

$$\left(\| g(t) \|^2 = \int_{-\infty}^{\infty} | g(t) |^2 \, dt \quad \text{and} \quad (g, h) = \int_{-\infty}^{\infty} g(t) \bar{h}(t) \, dt \right).$$

The combined operator $D_T B_W$ is the energy operator, and its eigenvalues are the energy eigenvalues studied by Landau and Pollak.

This paper concerns the generalization of these results on dimensionality to the class of functions representing near-end crosstalk transfer functions within a multipair cable. As an approximation of the coupling within a cable, it follows from the telegrapher's equation[3,4] that these transfer functions have the form

$$N(\omega) = i\omega \int_0^l e^{-2\Gamma(\omega) x} u(x) \, dx,$$

where $\Gamma(\omega) = i\beta(\omega) + \alpha(\omega)$ $(\alpha(\omega) \geqq 0)$ denotes the propagation function for a pair in the cable, l is the cable length, and the coupling function between two pair, $u(x)$, satisfies

$$\int_0^l | u(x) |^2 \, dx < \infty .$$

The physical meaning of $N(\omega)$ is specified in more detail in Section III. We wish to find the approximate dimensionality for the class of such functions either as viewed over some finite interval or, more generally, as weighted by some fixed square integrable function $F(\omega)$. Our approach, again, is to set up an energy operator, then study the energy eigenvalues to reach conclusions about the dimensionality.

The paper goes from the general to the particular. We first introduce a class of compact, integral operators and derive upper bounds for their eigenvalues. Next, we show that this class includes the energy operator corresponding to the crosstalk equation; this gives us bounds on the energy eigenvalues. From these bounds, one can draw quantitative conclusions about the dimensionality of the class of crosstalk functions.

II. A CLASS OF INTEGRAL OPERATORS

The first problem is to determine the distribution of eigenvalues for a special class of integral operators on the space of square integrable functions, $L^2(0, l)$. We characterize these operators by kernels of the form

$$K(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} | F(\omega) |^2 P(x, \omega) P^*(y, \omega) \, d\omega \qquad 0 \leqq x, y \leqq l,$$

where $F(\omega)$ is bounded and square-integrable, and $P(x, \omega)$ has the following two properties:

(*i*) The function $P(x, \omega)$ is bounded and the integral $\int_{\infty}^{\infty} \int_0^l |P(x, \omega)|^2 \, dx \, d\omega$ is bounded. As a consequence:
  (a) The ω-function, $U(\omega) \equiv \int_0^l P(x, \omega) u(x) \, dx$ is square-integrable when $u(x)$ is square-integrable over $(0, l)$. We assume its norm is nonzero.
  (b) $U(\omega)$ has a Fourier transform.
  (c) The operator, B_Y, limiting the Fourier transform to the interval $(-Y, Y)$ can be applied to the functions $U(\omega)$.
(*ii*) For all $u(x)$ in $L^2(0, l)$,

$$\| F(\omega)(I - B_Y)[U(\omega)] \|_W \leqq \epsilon(Y) \| u(x) \|_X$$

and $\epsilon(Y) \to 0$ as $Y \to \infty$. [Note: $\| \cdot \|_W$ denotes the standard norm on $L^2(-\infty, \infty)$ and $\| \cdot \|_X$ on $L^2(0, l)$. Also, "I" denotes the identity operator on $L^2(-\infty, \infty)$, i.e., $I[G(\omega)] = G(\omega)$.]

Suppose K is the operator having the kernel $K(x, y)$ above; then, since the kernel is square-integrable jointly in x and y, K is compact; i.e., it has a sequence of eigenvalues, say λ_n, $n = 0, 1, \cdots$, which approach zero as n gets large (see Ref. 5, p. 264). We shall bound the λ_n in terms of the eigenvalues, say $\lambda_n^0(F, Y)$, of the compact operator denoted by M_Y which acts on functions in $L^2(-Y, Y)$ and has the kernel

$$M_Y(\eta, \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} | F(\omega) |^2 e^{i\omega(\eta-\xi)} \, d\omega.$$

These eigenvalues are more directly accessible and better studied than the λ_n, and when

$$F(\omega) = \begin{cases} 1 & | \omega | \leqq 2\pi W \\ 0 & | \omega | > 2\pi W \end{cases}$$

they are exactly the eigenvalues of the operator $D_Y B_W$ studied by Landau aud Pollak. Also included in our bounding expression will be the eigenvalues, μ_n, $n = 0, 1, \cdots$, of the compact integral operator L acting on functions in $L^2(0, l)$ and having kernel

$$L(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(x, \omega) P^*(y, \omega) \, d\omega.$$

[*Note:* Using Schwarz's inequality on the ω-integral above and then property "(a)" for $P(x, \omega)$, we conclude that $L(x, y)$ is jointly square-integrable in x and y, and this assures the compactness of the operator L.]

Theorem: For the eigenvalues λ_n, $\lambda_n^0(Y, F)$, and μ_n just defined, we have for each $n = 0, 1, \cdots$

$$\lambda_n \leqq \min \left(\inf_Y \left[(a_n \lambda_n^0(Y, F))^{\frac{1}{2}} + \epsilon(Y) \right]^2, b_n \mu_n \right),$$

where $0 \leqq a_n \leqq \mu_0$, $0 \leqq b_n \leqq \max_\omega | F(\omega) |^2$, a_n approaches zero as n becomes large.

Proof: The Weyl–Courant Lemma (See Ref. 6, p. 251) implies that

$$\lambda_n = \inf_{S_n} \left(\sup_{u \perp S_n} \frac{(Ku, u)}{(u, u)} \right)$$

for any $n = 0, 1, 2, \cdots$, where S_n denotes an n-dimensional subspace of $L^2(0, l)$ and the infimum is taken over all such subspaces. Since (Lu, u) is nonnegative for all $u(x)$ in $L^2(0, l)$, we have

$$\lambda_n = \inf_{S_n} \sup_{u \perp S_n} \frac{(Ku, u)}{(Lu, u)} \frac{(Lu, u)}{(u, u)}.$$

So for all choices of S_n,

$$\lambda_n \leqq \sup_{u \perp S_n} \frac{(Ku, u)}{(Lu, u)} \sup_{u \perp S_n} \frac{(Lu, u)}{(u, u)}.$$

Choose S_n so as to minimize the latter factor. But the minimum value,

by tbe Weyl–Courant Lemma, is exactly tbe $n$tb eigenvalue of L. Thus, when S_n' is the appropriate subspace, then

$$\lambda_n \leqq \left(\sup_{u \perp S_{n'}} \frac{(Ku, u)}{(Lu, u)} \mu_{n'} \right) \equiv b_n \mu_n .$$

We claim $(Ku, u) = \| F(\omega) U(\omega) \|_w^2$ and $(Lu, u) = \| U(\omega) \|_w^2$ with

$$U(\omega) = \int_0^l P(x, \omega) u(x) \, dx,$$

from which it follows that

$$0 \leqq b_n \leqq \max_\omega | F(\omega) |^2 \quad \text{for all} \quad n.$$

To prove the claim, we have

$$\begin{aligned}
(Ku, u) &= \frac{1}{2\pi} \int_0^l u(y) \int_0^l u(x) \int_{-\infty}^{\infty} | F(\omega) |^2 P(x, \omega) P^*(y, \omega) \, d\omega \, dx \, dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} | F(\omega) |^2 \left(\int_0^l P(x, \omega) u(x) \, dx \right) \left(\int^l P^*(y, \omega) u(y) \, dy \right) d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} | F(\omega) U(\omega) |^2 \, d\omega = \| F(\omega) U(\omega) \|_w^2
\end{aligned}$$

and similarly for (Lu, u). (The integrand above is clearly absolutely integrable, so the conditions for the Fubini Theorem are met, and the order of integration can be switched freely.)

For tbe second part of the bound, tbe Minkowski inequality implies

$$(Ku, u) \leqq (\| F(\omega) B_Y[U(\omega)] \|_w + \| F(\omega)(I - B_Y)[U(\omega)] \|_w)^2.$$

So by assumption *(ii)* on the function $P(x, \omega)$ we get

$$\frac{(Ku, u)}{(u, u)} \leqq \left(\frac{\| F(\omega) B_Y[U(\omega)] \|_w}{\| u \|} + \epsilon(Y) \right)^2.$$

Now choose the subspace S_n so as to minimize the quantity

$$\sup_{u \perp S_n} \frac{\| F(\omega) B_Y[U(\omega)] \|_w}{\| B_Y[U(\omega)] \|_w}.$$

But this minimum value is upper bounded by the nth eigenvalue of the operator M_Y, defined previously, because the Weyl–Courant Lemma implies

$$\lambda_n^0(F, Y) = \inf_{R_n} \sup_{V \perp R_n} \frac{(M_Y V, V)_Y}{(V, V)_Y} ,$$

where R_n represents a subspace of $L^2(-Y, Y)$; and by the calculation above

$$(M_Y V, V) = \| F(\omega)\tilde{V}(\omega) \|_w^2 ,$$

where

$$\tilde{V}(\omega) = \int_{-Y}^{Y} e^{i\omega y} V(y)\, dy = B_Y[\tilde{V}(\omega)]$$

and the argument is finished by applying the Plancherel Theorem,

$$(V, V)_Y = [\tilde{V}(\omega), \tilde{V}(\omega)]_w .$$

Therefore, when S_n'' is the appropriate subspace of $L^2(0, l)$,

$$\lambda_n \leqq \sup_{u \perp S_n''} \frac{(Ku, u)}{(u, u)} \leqq [(a_n \lambda_n^0(F, Y))^{\frac{1}{2}} + \epsilon(Y)]^2,$$

where

$$a_n = \sup_{u \perp S_n''} \frac{\| B_Y[U(\omega)] \|_w^2}{\| u(x) \|_x^2}.$$

Note that

$$0 \leqq a_n \leqq \sup_u \frac{\| U(\omega) \|_w^2}{\| u(x) \|_x^2} = \left(\sup_u \frac{(Lu, u)}{(u, u)} \right) = \mu_0 .$$

Also, since L is compact, $a_n \to 0$ as $n \to \infty$. The inequality is good for all values of Y, so it is good for the infimum over Y. We have two upper bounds for the λ_n ; thus the minimum of the two is also an upper bound. This proves the theorem. Q.E.D.

III. APPLICATION TO CROSSTALK

The near-end crosstalk equation for multipair cable leads to an integral operator of the type in the theorem. The crosstalk transfer function $N(\omega)$ (in the frequency domain) is related to a coupling function along the cable, $u(x)$, by

$$N(\omega) = i\omega \int_0^l e^{-2\Gamma(\omega)x} u(x)\, dx,$$

where l denotes the cable length and $\Gamma(\omega)$ denotes the propagation function of a pair in the cable. A good approximation to $\Gamma(\omega)$ over the frequency range 0.1 to 10 mHz is

$$\Gamma(\omega) = \mathrm{k}_1 \sqrt{|\omega|} + i\, \mathrm{k}_2\, \omega + i \operatorname{sgn}(\omega)\, \mathrm{k}_1 \sqrt{|\omega|},$$

where the exact values of the constants k_1 and k_2 depend on the gage of the wire. More precisely, when a pair in a cable is excited by a signal with spectrum $G(\omega)$, the coupling within the cable will produce a signal with spectrum $G(\omega)N(\omega)$ at the near end of an unexcited pair. The energy of the crosstalk signal is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} | G(\omega)N(\omega) |^2 \, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} | \omega G(\omega) |^2 \left| \int_0^l e^{-2\Gamma(\omega)x} u(x)\, dx \right|^2 d\omega$$

By Fubini's Theorem [applicable when $\omega G(\omega) \in L^2(-\infty, \infty)$]we get,

$$\int_0^l u(y) \int_0^l u(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} | \omega G(\omega) |^2 e^{-2\Gamma(\omega)x - 2\Gamma^*(\omega)y} \, d\omega\, dx\, dy = (Ku, u).$$

Here K is a compact integral operator with the kernel

$$K(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} | \omega G(\omega) |^2 e^{-2\Gamma(\omega)x - 2\Gamma^*(\omega)y} \, d\omega.$$

We call K the energy operator and its eigenvalues, say, λ_n, $n = 0, 1, \cdots$, the energy eigenvalues. With a slightly more restrictive assumption on the function $G(\omega)$, we obtain for the λ_n the same bounds as before.

Proposition: When $(1 + | \omega |^q)\omega G(\omega)$ is in $L^2(-\infty, \infty)$ for $q > \frac{1}{4}$ and asymptotically for large ω, $Re(\Gamma(\omega)) \sim \omega^r$ with $r \geqq \frac{1}{2}$, then the previous bounds apply to the energy eigenvalues.

Proof: Let

$$P(x, \omega) = \frac{e^{-2\Gamma(\omega)x}}{1 + | \omega |^q};$$

then the previous properties assumed for $P(x, \omega)$ are satisfied. To demonstrate this: first, it is clear that $P(x, \omega)$ is bounded and it is tailored (i.e., the q values are just large enough) to be square-integrable jointly in x and w. Since $\int_0^l e^{-2zx} u(x)\, dx$ is analytic in z, it follows from the Plancherel Theorem that $U(\omega)$ has nonzero norm for all $u(x)$ in $L^2(0, l)$ with nonzero norm. Finally, using the Fubini Theorem,

$$\begin{aligned} B_Y\left[\int_0^l P(x, \omega)u(x)\, dx\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin Y(\eta - \omega)}{(\eta - \omega)} \int_0^l P(x, \omega)u(x)\, dx\, d\omega \\ &= \frac{1}{2\pi}\int_0^l u(x) \int_{-\infty}^{\infty} \frac{\sin Y(\eta - \omega)}{(\eta - \omega)} P(x, \omega)\, d\omega\, dx \\ &= \int_0^l B_Y[P(x, \omega)]u(x)\, dx; \end{aligned}$$

but $B_Y[P(x, \omega)]$ is uniformly bounded in ω for all values of x in the interval $[0, l]$, so

$$\rho_Y(x, \omega) \equiv (I - B_Y)[P(x, \omega)]$$

is square-integrable in x and for all ω,

$$| (I - B_Y)[U(\omega)] |^2 \leqq \int_0^l | \rho_Y(x, \omega) |^2 \, dx \int_0^l | u(x) |^2 \, dx.$$

Therefore,

$$\| F(\omega)(I - B_Y)[U(\omega)] \|_w \leqq \epsilon(Y) \, \| u(x) \|_x ,$$

where

$$\epsilon(Y) = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^l | F(\omega)\rho_Y(x, \omega) |^2 \, dx \, d\omega \right]^{\frac{1}{2}}.$$

Put $F(\omega) = (1 + | \omega |^q)\omega G(\omega)$; then

$$K(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} | F(\omega) |^2 \frac{e^{-2\Gamma(\omega)x}}{1 + | \omega |^q} \frac{e^{-2\Gamma^*(\omega)y}}{1 + | \omega |^q} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} | F(\omega) |^2 P(x, \omega) P^*(y, \omega) \, d\omega.$$

This is exactly the same form as before and so the energy eigenvalues have exactly the same bound. Q.E.D.

Corollary: When

$$F(\omega) = \begin{cases} 1 & | \omega | \leqq W \\ 0 & | \omega | > W \end{cases},$$

$$\lambda_n \leqq \min \left(\inf_Y \left[(\mu_0 \alpha_n(2YW))^{\frac{1}{2}} + \epsilon(Y, W) \right]^2, \mu_n \right),$$

where α_n denotes the eigenvalues associated with the prolate spheroidal waveforms (Cf. Ref. 2), and $\epsilon(Y, \omega)$ is $\epsilon(Y)$ with the W-dependence indicated.

Proof: Since $a_n \leqq \mu_0$, we can substitute μ_0 for a_n. The eigenvalues $\lambda_n^0(Y, F)$ in this special case are those studied by Landau and Pollak; they depend on the product of Y and W. Finally, $b_n \leqq 1$ in this case. Q.E.D.

IV. DIMENSIONALITY OF CROSSTALK FUNCTIONS

When approximating crosstalk functions over some interval (or as weighted by some square-integrable function) by linear combinations of functions, two practical questions arise. What is the most efficient set of functions, i.e., the one requiring the fewest number of functions to approximate any function in the class to a specified error tolerance? Then, for a given error tolerance, how many of these functions are required, i.e., what is the dimensionality?

There are at least two practical ways that information on crosstalk dimensionality can be used. For a specified error tolerance, the dimensionality gives the minimum number of independent measurements required to determine the crosstalk as a function of frequency. Thus it provides useful information to crosstalk measurement programs. Also, in efforts to reduce crosstalk over a specific range of frequencies, by subtracting linear combinations of fixed functions, the dimensionality indicates the minimum number of independent controls needed to meet a given criterion. Thus, dimensionality is a general concept, not tied to any particular method, either for measuring or for controlling crosstalk.

Before discussing dimensionality further, we answer the first question stated above. We show that the most efficient set is the set of eigenfunctions for the energy operator, i.e., the energy eigenfunctions. This result is a variant of Theorem I in Landau and Pollak's paper.[2] Our proof differs from theirs; and also we strive for the greatest generality by considering crosstalk functions multiplied by an arbitrary square-integrable function, $G(\omega)$. Later, in dealing with dimensionality, we shall take $G(\omega)$ as zero outside the finite frequency range of interest.

Theorem: The quantity

$$\sup_{N(\omega)} \min_{\{a_j\}_0^{J-1}} \left\| N(\omega)G(\omega) - \sum_{j=0}^{J-1} a_j N_j(\omega)G(\omega) \right\|_w^2 ,$$

where the supremum is taken over all crosstalk functions with normalized coupling function, is minimized by choosing $N_j(\omega)$, $j = 0, \cdots, J-1$, as the crosstalk functions with coupling functions $u_j(x)$ equal to the $(j+1)$th normalized energy eigenfunction (ordered according to decreasing eigenvalue).

Proof: Suppose $N_j(\omega)$, $j = 0, \cdots, J-1$, are linearly independent crosstalk functions and let P_J denote the projection operator onto the J-dimensional subspace in $L^2(-\infty, \infty)$ spanned by $\{G(\omega)N_j(\omega)\}_0^{J-1}$; put $P_J' = I - P_J$. Then,

$$\min_{\{a_j\}_0^{J-1}} \left\| N(\omega)G(\omega) - \sum_{j=0}^{J-1} a_j N_j(\omega)G(\omega) \right\|_w^2 = \| P'_J N(\omega)G(\omega) \|_w^2 .$$

Let A denote the operator taking coupling functions in $L^2(0, l)$ to $L^2(-\infty, \infty)$ such that

$$Au(x) = i\omega G(\omega) \int_0^l e^{-2\Gamma(\omega)x} u(x)\, dx = G(\omega)N(\omega).$$

(Note: If A^* denotes the adjoint of A, then $A^*A = K$, the energy operator.) The problem is to choose $u(x)$ (with $\| u \|_x = 1$) to maximize the quantity $\| P'_J Au(x) \|_w^2$ and then to choose the minimizing projection P'_J .

Since P'_J is a projection operator,

$$\| P'_J Au \|_w^2 = (P'_J Au, P'_J Au)_w = (A^* P'_J Au, u)_w$$

and the maximization over $u(x)$ gives the operator norm of the operator $A^* P'_J A$, i.e., the greatest eigenvalue. We denote this by $| A^* P'_J A |$. But

$$| A^* P'_J A | = |(A^* P'_J)(P'_J A)| = |(P'_J A)(A^* P'_J)| = | P'_J A A^* P'_J | ,$$

which follows from the more general result that BB^* and B^*B have the same nonzero eigenvalues when B is compact (Cf. Ref. 5, p. 262). The Weyl–Courant Lemma implies that $| P'_J A A^* P'_J |$ is minimized when P_J corresponds to the subspace spanned by the first J eigenfunctions of AA^* with the $(J + 1)$th eigenvalue as the minimal value. Since $K = A^*A$ and AA^* have the same nonzero eigenvalues, this minimal value is λ_J , i.e.,

$$\inf_{P_J} | A^* P'_J A | = \lambda_J .$$

When P_J is associated with the subspace spanned by $\{G(\omega)N_i(\omega)\}_0^{J-1}$, where the $N_i(\omega)$ are crosstalk functions with the first J energy eigenfunctions for coupling functions, then clearly

$$| A^* P'_J A | = \lambda_J .$$

Thus, this set of functions is most efficient. Q.E.D.

We note that the supremum in the theorem has been taken over all $N(\omega)$ (with associated coupling function having unit norm, i.e., $\int_0^l | u(x) |^2 dx = 1$). One can show that the suitable approximating functions are the same even if the supremum is taken over $N(\omega)$ with the additional constraint, $\| N(\omega)G(\omega) \|_w = b$ for a fixed value of b.

The main issue is dimensionality. Again, let $N(\omega)$ be an arbitrary crosstalk function. Suppose one wishes to approximate $N(\omega)G(\omega)$ by a

linear combination of fixed linearly independent functions to within a mean square error of δ^2. The preceding theorem indicates that, for greatest efficiency, one should use the energy eigenfunctions $N_j(\omega)$. The dimensionality relative to $G(\omega)$ is the smallest integer D such that

$$\min_{\{a_j\}} \left\| G(\omega)N(\omega) - \sum_{j=0}^{D-1} a_j G(\omega)N_j(\omega) \right\|_w^2 \leqq \delta^2$$

for all $N(\omega)$ with normalized coupling function. In terms of the energy eigenvalues this means

$$\lambda_D \leqq \delta \quad \text{but} \quad \lambda_{D-1} > \delta.$$

This is so because for any such function $N(\omega)$ there are coefficients $\{b_j\}$ such that

$$N(\omega) = \sum_{j=0}^{\infty} b_j N_j(\omega) \quad \text{and} \quad \sum_{j=0}^{\infty} | b_j^2 | = 1.$$

Thus,

$$\min_{\{a_j\}} \left\| G(\omega)N(\omega) - \sum_{j=0}^{D-1} a_j G(\omega)N_j(\omega) \right\|_\omega^2 = \left\| \sum_{j=D}^{\infty} G(\omega) b_j N_j(\omega) \right\|_\omega^2$$

$$= \sum_{j=D}^{\infty} b_j^2 \lambda_j^2 .$$

Since the eigenvalues λ_j are in descending order, this quantity is maximized by putting $b_j = \{{}^{0\ j \neq D}_{1\ j=D}$ and, therefore, it must be less than δ^2 for all choices of $\{b_j\}$.

Now we wish to work out the dimensionality for the practical case where one is concerned with a fixed frequency interval, $(-W/2\pi, W/2\pi)$. We modify the assumption of the corollary in Section III to say that

$$(1 + | k_2\omega |^q)\, \omega G(\omega) = \begin{cases} 1 & | \omega | \leqq W \\ 0 & | \omega | \geqq W \end{cases}.$$

The corollary indicates that the dimensionality (relative to δ) is upper bounded by the smallest integer D such that

$$(((\mu_0 \alpha_D(2YW))^{\frac{1}{2}} + \epsilon(Y, W))^2, \mu_D) < \delta \quad \text{(for some } Y\text{)}$$

and the modification means only that ϵ and the μ_D undergo a corresponding change. The eigenvalues $\alpha_D(2WY)$ are tabulated and plotted in Ref. 7. They decrease rapidly for increasing D greater than the

threshold value, $2C/\pi \equiv 2/\pi[2YW]$, as indicated there. This is generalized in Ref. 8. What must be calculated is the behavior of the eigenvalue sequence, $\{\mu_D\}$, and the function $\epsilon(Y, W)$.

Assume $q = 2$ and

$$\Gamma(\omega) = k_1(\sqrt{|\omega|} + \operatorname{sgn}(\omega) i \sqrt{|\omega|}) + ik_2\omega.$$

Though we leave k_2 unassigned, we assume relative to mile units, $k_1^2 = k_2$. (A typical value of k_2 is 8.0×10^6 second/mile for 19-gage wire.) An upper bound for $\epsilon(Y, W)$ is $\epsilon(Y, \infty)$ and naturally it is tighter for larger values of W. Explicitly,

$$\epsilon^2(Y, W) \leqq \frac{1}{2\pi} \int_0^l \int_{-\infty}^{\infty} \left| (I - B_Y)\left(\frac{e^{-2\Gamma(\omega)x}}{1 + (k_2\omega)^2}\right) \right|^2 d\omega\, dx.$$

To estimate this, note that the Fourier transform of $e^{-2\Gamma(\omega)x}/1 + (k_2\omega)^2$ is the convolution of the transforms for each factor taken separately, each of which is standard. Denote this by $f(y, x)$; then

$$f(y, x) = \int_{-\infty}^{y_0} \frac{1}{4k_2} e^{-|z|/k_2} \frac{xk_1}{\sqrt{\pi}} (y_0 - z)^{-\frac{3}{2}} \exp\left(-\frac{2x^2k_1^2}{y_0 - z}\right) dz,$$

where $y_o = y - 2k_2x$. By the Plancherel theorem, we have

$$\epsilon^2(Y, W) \leqq \int_0^l \int_{|y|>Y} |f(y, x)|^2\, dy\, dx.$$

To arrive at specific bounds on $\epsilon(Y, W)$, we bound the convolution and then perform the y and x integrations on this bound. The derivation appears in the Appendix; the result is

$$\epsilon^2(Y, W) \leqq \frac{l}{k_2}\left[\frac{e^{-2P}}{64} + \frac{3e^{-2P/3}}{64} + \frac{9l^2}{64\pi P^2} + \min\left\{\frac{9le^{-P/3}}{32P^{\frac{3}{2}}}, \frac{3le^{-P/3}}{8P^{\frac{1}{2}}}\right\}\right],$$

where $P = (Y - 2k_2l)/k_2$.

The remaining unspecified quantities in the bounding expression for the energy eigenvalues, λ_n, are the eigenvalues, μ_n, for the operator L. Note first that these depend on the two parameters, k_2 and l. The former is easily handled:

$$L(x, y) = \int_{-\infty}^{\infty} \frac{e^{-2\Gamma(\omega)x}}{1 + (k_2\omega)^2} \frac{e^{-2\Gamma^*(\omega)y}}{1 + (k_2\omega)^2} \frac{d\omega}{2\pi} \qquad 0 \leqq x, \quad y \leqq l.$$

Put $k_2\omega = \xi$; then we obtain

$$L(x, y) = \frac{2}{k_2} \int_0^{\infty} e^{-2(x+y)\sqrt{\xi}} \cos\left[2(\xi + \sqrt{\xi})(x - y)\right] \frac{d\xi}{(1 + \xi^2)^2}$$

and so the k_2-dependence is completely specified. The l-dependence, on the other hand, is not so easily isolated. The most important element of the sequence, $\{\mu_n\}$, is μ_o because it appears in conjunction with $\epsilon(Y, W)$. In fact,

$$\lambda_n \leqq \mu_0[(\alpha_n(2YW))^{\frac{1}{2}} + \epsilon(Y, W)/\mu_0^{\frac{1}{2}}]^2 \quad \text{for all} \quad Y.$$

Thus knowledge of μ_o gives us one completely specified bounding expression for each λ_n. At this point, the question is whether this bound is better or worse than the μ_n for a given n. We do not calculate the μ_n here and leave this question open. Rather we shall study the former expression in an example.

Let $l = 0.1$ mile. In this case, we have done a computer calculation for μ_o; the result is $1.57(k_2)^{-1}$. The problem is to find the value of Y which minimizes the bound for a given W and k_2. For a fixed W, the first term, $\sqrt{\alpha_n(2YW)}$, is reduced by decreasing Y, and $\epsilon(Y, W)$ is reduced by increasing Y; therefore, the best Y is some compromise value. Let $W = (m\pi/k_2l)$ and $Y = sk_2l$ for $m > 0$ and $s > 2$; then

$$\lambda_n \leqq \mu_0\Bigg[(\alpha_n(2\pi sm))^{\frac{1}{2}} + \left(\frac{1}{15.7}\left(\frac{e^{-2P}}{64} + \frac{3e^{-2P/3}}{64} + \frac{3}{64(s-2)^2} + \frac{3e^{-P/3}}{8\sqrt{10(s-2)}}\right)\right)^{\frac{1}{2}}\Bigg]^2.$$

For the first eigenvalue, ($n = 0$) when $\pi m \geqq 1$, then $\alpha_0(2\pi sm) \approx 1$ since $s > 2$ (Cf. Ref. 7, Fig. 2); consequently, μ_0 is the best bound available here. But when, for example, $\pi m = 0.25$ and s is chosen as 2.5, then

$$(\alpha_0(2\pi sm))^{\frac{1}{2}} \approx 0.8 \qquad \text{(Cf. Ref. 7, Fig. 2)}$$

and

$$\lambda_0 \leqq \mu_0[0.80 + 0.17]^2 \approx \mu_0(0.94) < \mu_0 .$$

In this case, it behooves us to use the more complicated bound. For 19-gage wire, this choice of m corresponds to a highest frequency of about 5×10^4 Hz.

For the tenth eigenvalue ($n = 9$) when $\pi m = 1$, choose $s = 5.5$; then $(\alpha_0(2\pi sm))^{\frac{1}{2}} \approx 0.99$ and

$$\lambda_9 \leqq \mu_0[0.09 + 0.09]^2 \approx 0.032\, \mu_0 .$$

This is in the vicinity of the best choice for s. This means that for $W = (1/k_2l)$ and $l = 0.1$ mile, the dimensionality of the crosstalk

functions relative to an error criterion, $0.032\ \mu_0$, is at most 10. It may be less than 10, but one needs lower bounds on the eigenvalues to determine that. Our technique does not carry over in any obvious way to a determination of lower bounds.

V. CONCLUSIONS

We have calculated bounds on the energy eigenvalues to determine the dimensionality of the crosstalk functions. Our analysis uses ideas developed by Landau and Pollak,[2] but our problem has a different character. Since any crosstalk function approaches zero for increasing frequency at a certain minimal rate, independent of the coupling function $u(x)$, the eigenvalues are insensitive to increases in the bandwidth W after a certain point, i.e., they saturate. This is indicated in the bounding expression by the presence of the μ_n which are independent of W. Hence, the question of which part of the bound is better depends on W: if W is large enough, the λ_n will have nearly saturated to μ_n and these are better, whereas smaller W-values are better handled by the more complicated expression which is sensitive to changes in W. This phenomena came up in our example for $n = 0$.

The dimensionality, as we have seen, presumes an error criteria. Given this, one can calculate from the bound, in any specific case, an upper bound on the dimensionality. The significance of this in a measurement program or in a crosstalk control scheme is to provide a realistic goal for reducing the number of independent measurements or controls, respectively. Since we have not derived lower bounds for the λ_n, the tightness of the upper bounds remains in question. Thus, the possibility of achieving greater reductions than our results would indicate is open.

VI. ACKNOWLEDGMENTS

It is a pleasure to acknowledge N. A. Strakhov for the original idea of generalizing the famous work of Landau and Pollak to dissipative systems. Also I am grateful to H. J. Landau for the benefit of much valuable criticism in the course of this work.

APPENDIX

We wish to bound the function

$$f(y, x) = \frac{x\mathrm{k}_1}{4\mathrm{k}_2\sqrt{\pi}} \int_{-\infty}^{y_0} e^{-|z|/\mathrm{k}_2}(y_0 - z)^{-\frac{3}{2}} \exp\left(-\frac{2x^2\mathrm{k}_1^2}{y_0 - z}\right) dz$$

by a more convenient function of y and x, both when $y > Y$ and when $y < -Y$. First suppose $y_o \geqq 0$; then

$$\int_{-\infty}^{y_0/3} e^{-|z|k_2}(y_0 - z)^{-\frac{3}{2}} \exp\left(-\frac{2x^2k_1^2}{y_0 - z}\right) dz$$
$$\leqq y_0^{-\frac{3}{2}}\left(\frac{2}{3}\right)^{-\frac{3}{2}} \int_{-\infty}^{\infty} e^{-|z|/k_2}\, dz = 3k_2 y_0^{-\frac{3}{2}}\left(\frac{3}{2}\right)^{\frac{1}{2}}.$$

Also,

$$\int_{y_0/3}^{y_0} e^{-z/k_2}(y_0 - z)^{-\frac{3}{2}} \exp\left(-\frac{2x^2k_1^2}{y_0 - z}\right) dz$$
$$\leqq e^{-y_0/3k_2} \int_0^{\infty} \gamma^{-\frac{1}{2}} \exp\left(-2x^2k_1^2\gamma\right) d\gamma = e^{-y_0/3k_2} \frac{(\pi/2)^{\frac{1}{2}}}{xk_1},$$

where we have put $\gamma = 1/(y_o - z)$ in the change of variables. Thus

$$f(y, x) \leqq \frac{3xk_1}{4}\left(\frac{3}{2\pi}\right)^{\frac{1}{2}} y_0^{-\frac{3}{2}} + \frac{1}{4k_2\sqrt{2}} e^{-y_0/3k_2}.$$

Next, suppose $y_o = y_0'$ with $y_0' > 0$. Then, after changing z to $-z$ in the integral above, we have

$$f(y, x) \leqq \frac{xk_1}{4k_2\sqrt{\pi}} e^{-y_0'/k_2} \int_{y_0'}^{\infty} (z - y_0')^{-\frac{3}{2}} \exp\left(-\frac{2x^2k_1^2}{z - y_0'}\right) dz$$
$$\leqq \frac{1}{4k_2\sqrt{2}} e^{-|y_0|/k_2}.$$

The details of this calculation parallel the former case.

Since

$$\epsilon^2(Y, W) \leqq \int_0^l \int_{|y|>Y} |\, f(y, x)\, |^2\, dy\, dx,$$

we can bound $\epsilon^2(Y, W)$ by performing the integrations on the above bounds for $f(y, x)$. First we have

$$\int_{|y|>Y} |\, f(y, x)\, |^2\, dy \leqq \frac{1}{k_2}\left[\frac{e^{-2P(x)}}{64} + \frac{3e^{-2P(x)/3}}{64} + \frac{27x^2}{64\pi P(x)^2}\right.$$
$$\left. + \min\left\{\frac{9xe^{-P(x)/3}}{16(P(x))^{\frac{3}{2}}}, \frac{3xe^{-P(x)/3}}{4(P(x))^{\frac{1}{2}}}\right\}\right],$$

where $P(x) = (Y - 2k_2x)/k_2$. In the latter term we have upper bounded $3/\pi$ by 1 and have put $(Y - 2k_2x)$ for y_o (in the former case before

the y-integration and in the latter after one integration by parts). Before doing the x-integration, replace $P(x)$ by $P \equiv P(l)$, thus obtaining a greater bound; then the x-integration gives us

$$\epsilon^2(Y, W) \leqq \frac{l}{\mathrm{k}_2}\left[\frac{e^{-2P}}{64} + \frac{3e^{-2P/3}}{64} + \frac{9l^2}{64\pi P^2} + \min\left\{\frac{9le^{-P/3}}{32P^{\frac{3}{2}}}, \frac{3le^{-P/3}}{8P^{\frac{1}{2}}}\right\}\right].$$

We shall always assume that $Y > 2\mathrm{k}_2 l$, i.e., that $P > 0$.

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